In this lecture we assume that there are no function symbols in the signature.
Quantifier rank of a formula

$QR(\varphi)$ is defined as follows:

$QR(\varphi \rightarrow \psi) = \max\left(QR(\varphi), QR(\psi)\right)$

$QR(\forall x \varphi) = 1 + QR(\varphi)$
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- $QR(\bot) = QR(t_1 = t_2) = QR(r(t_1, \ldots, t_n)) = 0$ for terms $t_1, \ldots, t_n$ and $r \in \Sigma_n^R$. 

$QR(\varphi \to \psi) = \max(QR(\varphi), QR(\psi))$.

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Formally: $QR$ is the nesting depth of quantifiers in the formula.
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Informally: $QR$ is the nesting depth of quantifiers in the formula.
Let $\mathcal{A}$ be a relational structure
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Then $\mathcal{A}_B$ is a structure over the signature $\Sigma$ of $\mathcal{A}$:
- the universe of $\mathcal{A}_B$ is $B$
- for $r \in \Sigma^R$ we define $r_{\mathcal{A}_B} := r_{\mathcal{A}} \cap B^n$. 
A, B – relational structures over Σ.
Nonempty subsets $A' \subseteq A$ and $B' \subseteq B$. 
\( \mathfrak{A}, \mathfrak{B} \) – relational structures over \( \Sigma \).

Nonempty subsets \( A' \subseteq A \) and \( B' \subseteq B \).

An isomorphism \( h : \mathfrak{A}_{|A'} \cong \mathfrak{B}_{|B'} \) of induced substructures is called a \underline{partial isomorphism} from \( \mathfrak{A} \) to \( \mathfrak{B} \).
Partial isomorphism

\[ A, B \] – relational structures over \( \Sigma \).

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An isomorphism \( h : A|_A' \cong B|_{B'} \) of induced substructures is called a partial isomorphism from \( A \) to \( B \).

Its domain is \( \text{dom}(h) = A' \), and range is \( \text{rg}(h) = B' \).
We adopt the convention that $\emptyset$ is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$ with empty domain and range.
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For two partial isomorphisms $g, h$ from $\mathcal{A}$ to $\mathcal{B}$ we write $g \subseteq h$ when $\text{dom}(g) \subseteq \text{dom}(h)$ and $g(a) = h(a)$ for all $a \in \text{dom}(g)$; alternatively, when $g$ is included in $h$ as a set.
Let \( m \in \mathbb{N} \).

Structures \( \mathcal{A} \) and \( \mathcal{B} \) are \( m \)-isomorphic (denoted \( \mathcal{A} \cong_m \mathcal{B} \)), if there exists a family \( \{ I_n \mid n \leq m \} \) such that:

1. Each \( I_n \) is a nonempty set of partial isomorphisms from \( \mathcal{A} \) to \( \mathcal{B} \).
2. For each \( h \in I_{n+1} \) and each \( b \in \mathcal{B} \), there exists \( g \in I_n \) such that \( h \subseteq g \) and \( b \in \text{rg}(g) \).
3. For each \( h \in I_{n+1} \) and each \( a \in \mathcal{A} \), there exists \( g \in I_n \) such that \( h \subseteq g \) and \( a \in \text{dom}(g) \).
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Structures $A$ and $B$ are $m$-isomorphic (denoted $A \cong_m B$), if there exists a family $\{I_n \mid n \leq m\}$ such that:

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- **Back** For each $h \in I_{n+1}$ and each $b \in B$ there exists $g \in I_n$ such that $h \subseteq g$ and $b \in \text{rg}(g)$. 
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- **Forth** For each $h \in I_{n+1}$ and each $a \in A$ there exists $g \in I_n$ such that $h \subseteq g$ and $a \in dom(g)$. 
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Structures \( \mathcal{A} \) and \( \mathcal{B} \) are \( m \)-isomorphic (denoted \( \mathcal{A} \cong_m \mathcal{B} \)), if there exists a family \( \{I_n \mid n \leq m\} \) such that:

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The family \( \{I_n \mid n \leq m\} \) is called an \( m \)-isomorphism of \( \mathcal{A} \) and \( \mathcal{B} \), denoted \( \{I_n \mid n \leq m\} : \mathcal{A} \cong_m \mathcal{B} \).
Finite isomorphism

Two structures \( A, B \) are finitely isomorphic, (denoted \( A \cong_{\text{fin}} B \)) if there exists a family \( \{ I_n \mid n \in \mathbb{N} \} \), whose each subfamily \( \{ I_n \mid n \leq m \} \) is an \( m \)-isomorphism.
Finite isomorphism

Two structures $\mathcal{A}$, $\mathcal{B}$ are **finitely isomorphic**, (denoted $\mathcal{A} \cong_{\text{fin}} \mathcal{B}$) if there exists a family $\{I_n \mid n \in \mathbb{N}\}$, whose each subfamily $\{I_n \mid n \leq m\}$ is an $m$-isomorphism.

If $\{I_n \mid n \leq m\}$ has the above property, we write $\{I_n \mid n \leq \mathbb{N}\} : \mathcal{A} \cong_{\text{fin}} \mathcal{B}$

This family is called a **finite isomorphism**.
Finite isomorphism

Fakt
Finite isomorphism

Fakt

If $A \cong B$, then $A \cong_{\text{fin}} B$. 
Finite isomorphism

Fakt

- If $A \cong B$, then $A \cong_{fin} B$.
- If $A \cong_{fin} B$ and the universe $A$ of $A$ is finite, then $A \cong B$. 
Proof: See blackboard.
Elementary equivalence

*Repetitio est mater studiorum*

A and B are elementary equivalent (denoted \( A \equiv B \)), if for each sentence \( \varphi \) of first-order logic
\( A \models \varphi \) iff \( B \models \varphi \).
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A and B are elementary equivalent (denoted $A \equiv B$), if for each sentence $\varphi$ of first-order logic

$A \models \varphi$ iff $B \models \varphi$.

A and B are $m$-elementary equivalent (denoted $A \equiv_m B$), if for each sentence $\varphi$ of quantifier rank at most $m$ holds

$A \models \varphi$ iff $B \models \varphi$. 
Fact

$\mathcal{A} \cong_{fin} \mathcal{B}$ if and only if for every natural $m$ holds $\mathcal{A} \equiv_m \mathcal{B}$. 
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$\mathcal{A} \cong_{fin} \mathcal{B}$ if and only if for every natural $m$ holds $\mathcal{A} \cong_m \mathcal{B}$.

Proof:

Suppose that for each $m$ there exists $\{l^m_n \mid n \leq m\}$ as in the definition of $\cong_m$.

The family $\{J_n \mid n \in \mathbb{N}\}$ defined by

$$J_n = \bigcup_{m \in \mathbb{N}} l^m_n$$

satisfies the definition of $\cong_{fin}$.
Theorem [Fraïssé]
Let $\Sigma$ by a finite relational signature;
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Theorem 1 (Fraïssé’s Theorem). The second equivalence follows from the first one. We prove the first one from left to right.
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Proof of Fraïssé’s Theorem

The second equivalence follows from the first one. We prove the first one from left to right. We fix $m \in \mathcal{N}$. 
Induction thesis

Let

\[ \{ I_n | n \leq m \} : A \sim = m \]

\( g \in I_n \)

\( \phi \) is a formula

\( FV(\phi) = x_1, \ldots, x_r \)

\( QR(\phi) \leq n \leq m \)

The following are equivalent:

\[ A, x_1 : a_1, \ldots, x_r : a_r \mid = \phi \]

\[ B, x_1 : g(a_1), \ldots, x_r : g(a_r) \mid = \phi. \]
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\[ \{ l_n \mid n \leq m \} : A \cong_m B \]
Induction thesis

Let

- \{l_n \mid n \leq m\} : \mathcal{A} \cong_m \mathcal{B}
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- \( \{ l_n \mid n \leq m \} : A \cong_m B \)
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- \( \varphi \) be a formula
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  - \( QR(\varphi) \leq n \leq m \)

The for each \( a_1, \ldots, a_r \in \text{dom}(g) \) the following are equivalent:

\[
\mathcal{A}, x_1 : a_1, \ldots, x_r : a_r \models \varphi
\]

\[
\mathcal{B}, x_1 : g(a_1), \ldots, x_r : g(a_r) \models \varphi.
\]
Induction

For atomic formulas the thesis follows from the fact that $g$ is a partial isomorphism. If $\phi \rightarrow \psi \rightarrow \xi$, then the following are equivalent:

$\begin{align*}
A, x_1: a_1, \ldots, x_r: a_r | = \psi \rightarrow \xi & \quad \text{or} \quad A, x_1: a_1, \ldots, x_r: a_r \not| = \psi \\
B, x_1: g(a_1), \ldots, x_r: g(a_r) | = \psi \rightarrow \xi & \quad \text{or} \quad B, x_1: g(a_1), \ldots, x_r: g(a_r) \not| = \psi
\end{align*}$
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- For atomic formulas, the thesis follows from the fact that $g$ is a partial isomorphism.
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- $\mathcal{B}, x_1: g(a_1), \ldots, x_r: g(a_r) \not\models \psi$ or $\mathcal{B}, x_1: g(a_1), \ldots, x_r: g(a_r) \models \xi$
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- $\mathcal{B}, x_1: g(a_1), \ldots, x_r: g(a_r) \models \psi \rightarrow \xi$
Let \( \varphi \) be \( \forall x_r + 1 \psi \).

By assumption, \( \text{QR} (\varphi) \leq n \), we get \( \text{QR} (\psi) \leq n - 1 \).

The following are equivalent:

\( (A, x_1 : a_1, \ldots, x_r : a_r) | = \varphi \)

For all \( a \in A \) holds

\( (A, x_1 : a_1, \ldots, x_r : a_r, x_{r+1} : a) | = \psi \)

For all \( a \in A \) there exists \( h \in I \) so that \( g \subseteq h \), \( a \in \text{dom} (h) \) and

\( (B, x_1 : g(a_1), \ldots, x_r : g(a_r), x_{r+1} : h(a)) | = \psi \)

For all \( b \in B \) holds

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For all $a \in A$ holds $(A, x_1): a_1, \ldots, x_r: a_r, x_{r+1}: a)$

For all $a \in A$ there exists $h \in I_{n-1}$ so that $g \subseteq h$, $a \in \text{dom}(h)$ and $(A, x_1): a_1, \ldots, x_r: a_r, x_{r+1}: h(a))$

For all $b \in B$ there exists $h \in I_{n-1}$ so that $g \subseteq h$, $b \in \text{rg}(h)$ and $(B, x_1): g(a_1), \ldots, x_r: g(a_r), x_{r+1}: h(b))$
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    $(\mathcal{A}, x_1 : a_1, \ldots, x_r : a_r, x_{r+1} : a) \models \psi$
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3. For all $a \in A$ there exists $h \in l_{n-1}$ so that $g \subseteq h$, $a \in \text{dom}(h)$ and $(\mathcal{A}, x_1 : a_1, \ldots, x_r : a_r, x_{r+1} : a) \models \psi$
4. For all $a \in A$ there exists $h \in l_{n-1}$ so that $g \subseteq h$, $a \in \text{dom}(h)$ and $(\mathcal{B}, x_1 : g(a_1), \ldots, x_r : g(a_r), x_{r+1} : h(a)) \models \psi$
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- $(B, x_1 : g(a_1), \ldots, x_r : g(a_r)) \models \varphi$. 
**Fakt**

If $\mathcal{A}, \mathcal{B}$ are two finite linear orders of cardinalities $\geq 2^m$, then $\mathcal{A} \equiv_m \mathcal{B}$. 
Proof

Without loss of generality let ...
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Proof

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- $2^m < N \leq M$. 
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Wykazujemy, że \( A \approx_m B \).
Without loss of generality let

- \( A = \{0, \ldots, N\} \),
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- \( 2^m < N \leq M \).

Wykazujemy, że \( A \cong_m B \).

For \( k \leq m \) we define “distance” \( d_k \) between elements by

\[
d_k(a, b) = \begin{cases} 
|b - a| & \text{if } |b - a| < 2^k \\
\infty & \text{otherwise}
\end{cases}
\]
Proof

Without loss of generality let

- $A = \{0, \ldots, N\}$,
- $B = \{0, \ldots, M\}$,
- $2^m < N \leq M$.

Wykazujemy, że $\mathcal{A} \cong_m \mathcal{B}$.
For $k \leq m$ we define “distance” $d_k$ between elements by

$$d_k(a, b) = \begin{cases} |b - a| & \text{if } |b - a| < 2^k \\ \infty & \text{otherwise} \end{cases}$$

Now see blackboard.
The Ehrenfeucht Game $G_m(A, B)$ is played by two players: I and II (Spoiler and Duplicator, Adam and Eve, Samson and Delilah, ...)

In the $i$-th round ($i = 1, \ldots, m$) the players make their moves:

Player I chooses:
- one of the structures
- an element of its universe (denoted $a_i$ if from $A$, $b_i$ if from $B$)

Player II chooses:
- the other structure
- an element of its universe (denoted $a_i$ if from $A$, $b_i$ if from $B$)

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$A, B$ – structures over $\Sigma$, additionally $A \cap B = \emptyset$. 


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And the winner is.

In the round the chosen elements are \(a_1, \ldots, a_m \in A\) and \(b_1, \ldots, b_m \in B\).

Player II wins if the mapping \(h = \{\langle a_i, b_i \rangle | i = 1, \ldots, m\}\) is a partial isomorphism from \(A\) to \(B\).

Otherwise Player I wins.

Player II has a winning strategy in \(G_m(A, B)\), if he/she can win any play, irrespective of the moves of Player I.
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Theorem [Ehrenfeucht]

Player I has a winning strategy in $G(A, B)$ if and only if $A \sim_m B$.

Player I has a winning strategy in $G_m(A, B)$ for each $m$ if and only if $A \sim_n B$. 
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Theorem If \( \mathcal{A} = \langle A, \leq^A \rangle \) and \( \mathcal{B} = \langle B, \leq^B \rangle \) are both linear orders without maximal and minimal elements then \( \mathcal{A} \equiv \mathcal{B} \).

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Proof: See blackboard
There is no sentence of first-order logic which distinguishes continuous linear orders from noncontinuous ones.

\[\langle \mathbb{R}, \leq \rangle \equiv \langle \mathbb{Q}, \leq \rangle.\]
Corollary

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Theories complete theories

Theory is a set of sentences closed under semantical consequence, i.e., set $\Delta$ such that $\Delta |\models \varphi$ holds only when $\varphi \in \Delta$.

Examples of theories:

$\{ \varphi \mid \Gamma \models \varphi \}$, called an axiomatic theory

$\text{Th}(K) = \{ \varphi \mid A |\models \varphi, \text{ dla } A \in K \}$ (theory of a class $K$ of structures)

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