First-order logic – syntax and semantics

First-order logic is also known as predicate logic and predicate calculus

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If the signature is finite and the arities are known, it is often presented as a sequence of symbols, e.g., $+, \cdot, 0, 1$

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- For each two terms t_1, t_2, the expression $(t_1 = t_2)$ is an atomic formula

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- If φ, ψ are formulas, then $(\varphi \rightarrow \psi)$ is a formula
- If φ is a formula and $x \in X$ is a variable, then $(\forall x \varphi)$ is a formula

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A formula without quantifiers is an open formula.

A formula without free variables is a sentence, or a closed formula.

Syntax abbreviations

Additional propositional connectives are abbreviations:

- $(\neg\varphi)$ for $\varphi \to \bot$
- $(\varphi \vee \psi)$ for $((\neg\varphi) \to \psi)$
- $(\varphi \wedge \psi)$ for $(\neg((\neg\varphi) \vee (\neg\psi)))$
- $(\varphi \leftrightarrow \psi)$ for $((\varphi \to \psi) \wedge (\psi \to \varphi))$

The existential quantifier is an abbreviation, too:

$$(\exists x \varphi) \qquad \text{means} \qquad (\neg(\forall x \neg\varphi)).$$

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The distinction between free and bound variables resembles the distinction between local and global variables in a procedure

Semantics of formulas

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Notation: $\mathfrak{A} = \langle A, f_1^{\mathfrak{A}}, \ldots, f_n^{\mathfrak{A}}, r_1^{\mathfrak{A}}, \ldots, r_m^{\mathfrak{A}} \rangle$, where $f_1, \ldots, f_n, r_1, \ldots, r_m$ are the symbols in the signature

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For a valuation ϱ, a variable $x \in X$ and an element $a \in A$ we define a modified valuation $\varrho_x^a : X \to A$:

$$\varrho_x^a(y) = \begin{cases} \varrho(y) & y \neq x \\ a & \text{otherwise} \end{cases}$$

Values of terms

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- $[\![x]\!]^{\mathfrak{A}}_{\varrho} = \varrho(x)$.
- $[\![f(t_1, \dots, t_n)]\!]^{\mathfrak{A}}_{\varrho} = f^{\mathfrak{A}}([\![t_1]\!]^{\mathfrak{A}}_{\varrho}, \dots, [\![t_n]\!]^{\mathfrak{A}}_{\varrho})$.

Formula satisfaction

$(\mathfrak{A}, \varrho) \models \varphi$ is read:

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- $(\mathfrak{A}, \varrho) \models (\forall x \varphi)$ iff for every $a \in A$ holds $(\mathfrak{A}, \varrho_x^a) \models \varphi$.

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Fact

For any Σ-structure $\mathfrak{A}$ and any formula φ, if valuations ϱ and ϱ' assign equal values to all free variables od φ, then

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Hence simplified notation: $(\mathfrak{A}, x : a, y : b) \models \varphi$ instead of $(\mathfrak{A}, \varrho) \models \varphi$, when $\varrho(x) = a$ and $\varrho(y) = b$, and there are no other free variables in φ

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$$r^{\mathfrak{A}}(a_1, \ldots, a_n) \;\text{ iff }\; r^{\mathfrak{B}}(h(a_1), \ldots, h(a_n))$$

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- Identity $\mathrm{id}_A : A \to A$ is an isomorphism $\mathrm{id}_A : \mathfrak{A} \cong \mathfrak{A}$

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If $h : \mathfrak{A} \cong \mathfrak{B}$ then for every formula φ

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Isomorphisms and logic

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If $x_1, \ldots, x_n$ are the free variables of φ, then

$$(\mathfrak{A}, x_1 : a_1, \ldots, x_n : a_n) \models \varphi \text{ iff } (\mathfrak{B}, x_1 : h(a_1), \ldots, x_n : h(a_n)) \models \varphi$$

Isomorphisms and logic cntd.

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Elementary equivalence

$\mathfrak{A}$ and $\mathfrak{B}$ are elementary equivalent (denoted $\mathfrak{A} \equiv \mathfrak{B}$), iff for every sentence φ of first-order logic over their common signature, $\mathfrak{A} \models \varphi$ if and only if $\mathfrak{B} \models \varphi$

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Intuitively, isomorphic structures are logically indistinguishable

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Sentence φ is a tautology (denoted $\models \varphi$), if it is valid in every Σ-structure

The first proof: theorem

If $h : \mathfrak{A} \cong \mathfrak{B}$ then for every formula φ

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$$\begin{aligned} h([\![f(t_1, \ldots, t_n)]\!]^{\mathfrak{A}}_{\varrho} &= h(f^{\mathfrak{A}}([\![t_1]\!]^{\mathfrak{A}}_{\varrho}, \ldots, [\![t_n]\!]^{\mathfrak{A}}_{\varrho})) \\ &= f^{\mathfrak{B}}(h([\![t_1]\!]^{\mathfrak{A}}_{\varrho}), \ldots, h([\![t_n]\!]^{\mathfrak{A}}_{\varrho})) \\ &= f^{\mathfrak{B}}([\![t_1]\!]^{\mathfrak{B}}_{h \circ \varrho}, \ldots, [\![t_n]\!]^{\mathfrak{B}}_{h \circ \varrho}) \\ &= [\![f(t_1, \ldots, t_n)]\!]^{\mathfrak{B}}_{h \circ \varrho} \end{aligned}$$

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- $$\begin{aligned}(\mathfrak{A}, \varrho) \models r(t_1, \ldots, t_n) &\text{ iff } \langle [\![t_1]\!]^{\mathfrak{A}}_{\varrho}, \ldots, [\![t_n]\!]^{\mathfrak{A}}_{\varrho} \rangle \in r^{\mathfrak{A}} \\ &\text{ iff } \langle h([\![t_1]\!]^{\mathfrak{A}}_{\varrho}), \ldots, h([\![t_n]\!]^{\mathfrak{A}}_{\varrho}) \rangle \in r^{\mathfrak{B}} \\ &\text{ iff } \langle [\![t_1]\!]^{\mathfrak{B}}_{h \circ \varrho}, \ldots, [\![t_n]\!]^{\mathfrak{B}}_{h \circ \varrho} \rangle \in r^{\mathfrak{B}} \\ &\text{ iff } (\mathfrak{B}, h \circ \varrho) \models r(t_1, \ldots, t_n)\end{aligned}$$

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$$\begin{aligned}(\mathfrak{A}, \varrho) \models t_1 = t_2 &\text{ iff } [\![t_1]\!]^{\mathfrak{A}}_{\varrho} = [\![t_2]\!]^{\mathfrak{A}}_{\varrho} \\ &\text{ iff } h([\![t_1]\!]^{\mathfrak{A}}_{\varrho}) = h([\![t_2]\!]^{\mathfrak{A}}_{\varrho}) \\ &\text{ iff } [\![t_1]\!]^{\mathfrak{B}}_{h\circ\varrho} = [\![t_2]\!]^{\mathfrak{B}}_{h\circ\varrho} \\ &\text{ iff } (\mathfrak{B}, h \circ \varrho) \models t_1 = t_2\end{aligned}$$

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$\varphi(t/x)$ is the result of substituting t for every free occurrence of a variable x in φ.

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Substituting y for x in those formulas yields

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- otherwise the substitution is not permissible

Substitution lemma

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Then:

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where $a = [\![t]\!]^{\mathfrak{A}}_{\varrho}$.

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  $$[\![s(t/x)]\!]^{\mathfrak{A}}_{\varrho} = [\![s]\!]^{\mathfrak{A}}_{\varrho^a_x}$$

  where $a = [\![t]\!]^{\mathfrak{A}}_{\varrho}$.
- For any formula φ, if term t is permissible for x in φ, then

  $$(\mathfrak{A}, \varrho) \models \varphi(t/x) \text{ iff } (\mathfrak{A}, \varrho^a_x) \models \varphi,$$

  where $a = [\![t]\!]^{\mathfrak{A}}_{\varrho}$.